§7.1 Diagonalization of Symmetric Matrices

Definition

A matrix A is said to be symmetric if A is square and $A = A^T$

Examples

- Any square diagonal matrix is symmetric
- $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$
- etc.

Theorem

If A is a symmetric matrix, then any two eigenvectors of A corresponding to different eigenvalues are orthogonal. In other words, if

$$Av_1 = \lambda_1 v_1 \quad \text{and} \quad Av_2 = \lambda_2 v_2$$

with $\lambda_1 \neq \lambda_2$, then $v_1 \cdot v_2 = 0$

This allows us to diagonalize symmetric matrices in a special way. Before that let's prove the theorem.

Proof

Suppose A is symmetric $(A=A^T)$, $Av_1=\lambda_1 v_1$, $Av_2=\lambda_2 v_2$, and λ_1, λ_2 are different eigenvalues $(\lambda_1 \neq \lambda_2)$. Then to show $v_1 \cdot v_2 = 0$, notice

$$
\begin{aligned}
\lambda_1(v_1 \cdot v_2) &= \lambda_1 v_1 \cdot v_2 \\
&= (\lambda_1 v_1)^T v_2 \\
&= (Av_1)^T v_2 \\
&= (v_1^T A^T) v_2 \\
&= v_1^T A^T v_2 \\
&= v_1^T A v_2 \\
&= v_1^T (\lambda_2 v_2) \\
&= v_1 \cdot \lambda_2 v_2 \\
&= \lambda_2 (v_1 \cdot v_2)
\end{aligned}
$$

Thus $\lambda_1(v_1 \cdot v_2) = \lambda_2(v_1 \cdot v_2)$ so

$$\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0}(v_1 \cdot v_2) = 0$$

since $\lambda_1 \neq \lambda_2$ it follows $v_1 \cdot v_2 = 0$

Recall a square $n \times n$ matrix P is said to be <u>orthogonal</u> if P is invertible with $P^{-1} = P^T$.

From §6.2 this is equivalent to saying the columns of P form an orthonormal basis for $\mathbb{R}^n$.

<u>Definition</u>

An $n \times n$ matrix A is <u>orthogonally diagonalizable</u> if there exists an orthogonal matrix P and diagonal matrix D such that

$$A = PDP^T \qquad \text{(note that } P^T = P^{-1}\text{)}$$

In other words, there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

<u>Theorem</u>

An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric.

Example

Orthogonally diagonalize $A = \begin{bmatrix} 6 & 3 \\ 3 & 14 \end{bmatrix}$

solution: $\det(A - \lambda I) = \det\begin{bmatrix} 6-\lambda & 3 \\ 3 & 14-\lambda \end{bmatrix}$

$$= \lambda^2 - 20\lambda + 84 - 9$$

$$= \lambda^2 - 20\lambda + 75$$

$$= (\lambda - 5)(\lambda - 15)$$

eigenvalues

$\lambda_1 = 5$, $\lambda_2 = 15$

Eigenvectors:

$\lambda_1 = 5$: $A - 5I = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ $\quad v_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\lambda_2 = 15$: $A - 15I = \begin{bmatrix} -9 & 3 \\ 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

We see that $v_1 \cdot v_2 = 0$, but we need orthonormal eigenvectors. Since

$\|v_1\| = \sqrt{10}$ and $\|v_2\| = \sqrt{10}$

we can take

$u_1 = \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$

Thus

$$A = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 15 \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}^T$$

Remark

- Eigenvectors corresponding to <u>different</u> eigenvalues are orthogonal
- If you have an eigenvalue of multiplicity 2 or higher with multiple associated ~~eigenvalues~~ eigenvectors, they might not be orthogonal!
- In this case you can use the gram-schmidt Process.

Example

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$ has eigenvectors/eigenvalues

$\lambda_1 = 7$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$\lambda_2 = -2$

$$v_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

we see $v_1 \cdot v_3 = 0$ and $v_2 \cdot v_3 = 0$ but $v_1 \cdot v_2 \neq 0$. However by gram-schmidt (or just orthogonal projections)

- $$z_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- $$z_2 = v_2 - \left(\frac{v_2 \cdot z_1}{z_1 \cdot z_1}\right) z_1$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \left(\frac{-1+0+0}{1+0+1}\right)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 2 \\ 1/2 \end{bmatrix}$$

Thus

$$z_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad z_2 = \begin{bmatrix} -1/2 \\ 2 \\ 1/2 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

is a set of orthogonal eigenvectors.

Spectral Decomposition

Suppose A is an $n \times n$ symmetric matrix with orthonormal eigenvectors $u_1, \ldots, u_n$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Then we can write

$$A = PDP^T$$

$$= \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$= \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$$

The formula

$$A = \lambda_1 u_1 u_1^T + \cdots + \lambda_n u_n u_n^T$$

is called the <u>spectral decomposition</u> of A.

Example

From example 1, we were able to write

$$\begin{bmatrix} 6 & 3 \\ 3 & 14 \end{bmatrix} = \begin{array}{cc} \left[\begin{matrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{matrix}\right] \\ \begin{matrix} \quad u_1 \quad & \quad u_2 \quad \end{matrix} \end{array} \begin{array}{cc} \begin{matrix} \lambda_1 & \quad \end{matrix} \\ \left[\begin{matrix} 5 & 0 \\ 0 & 15 \end{matrix}\right] \\ \begin{matrix} \quad & \lambda_2 \end{matrix} \end{array} \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

The spectral decomposition of A is

$$A = 5\, u_1 u_1^T + 15\, u_2 u_2^T$$

$$= 5 \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \end{bmatrix} + 15 \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$= 5 \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix} + 15 \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}$$